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TWO REMARKS ON FOURIER–STIELTJES TRANSFORMS

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ABSTRACT

Special refinements of a theorem of Rajchman are given, along with some comments on approximation of certain Fourier–Stieltjes transforms.

We shall be concerned with two separate observations (which had their origin in one simple lemma). The second (§2) concerns refinements, in special cases, of the theorem of Rajchman [4] which states that a complex measure μ on the circle group T satisfies

$$\lim_{n\to+\infty}\hat{\mu}(n)=0 \tag{0}$$

iff the same is true with $|\mu|$ in place of μ, and so iff

$$\lim_{n\to-\infty}\hat{\mu}(n)=0.$$

The first (§1) is a comment on the question, for a locally compact abelian group G, of what can be said of those measures μ in $M(G)$ whose Fourier–Stieltjes transforms $\hat{\mu}$ approximate a fixed transform $\hat{\lambda}$ well?

§1. In view of the discontinuity of the map $\hat{\mu}\to\mu$, little should be expected in answer to the preceding question. Our purpose here is to point out one instance when something can in fact be said: if $\hat{\lambda}$ is bounded away from zero $\hat{\mu}$ will approximate $\hat{\lambda}$ well only if a certain proportion of their masses are mutually absolutely continuous (with the proportion depending on our bound and the degree of approximation). More precisely, we shall prove

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THEOREM 1. *Suppose $\hat{\lambda}$ and $\hat{\mu}$ are bounded away from zero and, on a subset F of the dual Γ of G containing some translate of each compact $K \subset G$, satisfy*

$$|\operatorname{sgn} \hat{\lambda} - \operatorname{sgn} \hat{\mu}| \leqq \theta < \sqrt{2}. \tag{1}$$

Then if λ_μ is the part of λ absolutely continuous with respect to $|\mu|$ we have

$$\|\hat{\lambda}^{-1}\hat{\lambda}_\mu\|_\infty \geqq \frac{1}{2}(\sqrt{2} - \theta) = \eta. \tag{2}$$

In fact, the same is true if F is the *generator of an invariant filter* in the sense of [3], viz., F lies in a dense subgroup Γ_0 of Γ and contains some translate of each finite subset of Γ_0; moreover $\hat{\lambda}$ and $\hat{\mu}$ need only be bounded away from zero on F if the left side of (2) is replaced by $\sup|\hat{\lambda}^{-1}\hat{\lambda}_\mu(F)|$.

In particular, (2) implies

$$\|\lambda_\mu\| \geqq \|\hat{\lambda}_\mu\|_\infty \geqq \eta / \|\hat{\lambda}^{-1}\|_\infty = \eta \inf|\hat{\lambda}|, \tag{2'}$$

but a stronger inequality on the norms of λ_μ and $\hat{\lambda}_\mu$ actually holds.

THEOREM 2. *Under the hypotheses of Theorem 1,*

$$\|\hat{\lambda}_\mu\| \geqq (1 - \theta^2/2)\inf|\hat{\lambda}_d| \geqq (1 - \theta^2/2)\inf|\hat{\lambda}(F)| \tag{3}$$

where λ_d is the discrete component of λ.

Since

$$1 - \theta^2/2 = \left(\frac{\sqrt{2} - \theta}{2}\right)(\sqrt{2} + \theta) = \eta(\sqrt{2} + \theta)$$

(3) is a considerable improvement over (2′). Note that the restriction that our transforms be bounded away from zero is essential: if μ is a singular measure on $\mathbf{R}$ and $f \in L_1(\mathbf{R})$ has $\hat{f} > 0$ then $\operatorname{sgn} \hat{\mu} \equiv \operatorname{sgn}(f * \mu)^\wedge$. In fact (2) and (3) result precisely from overlap of the discrete components of our measures.

The proofs of both results are related, and are consequences of conversations with Yitzhak Katznelson and Don Marshall, to whom I would like to express my thanks. Both depend on the fact that the mean value of a Fourier–Stieltjes transform is the mass of measure at the identity [5], and that mean values are determined on sets such as F, in particular on generators of invariant filters [3, lemma 1], viz., for weakly almost periodic functions f and g on Γ, if $f \geqq g$ on F then $M(f) \geqq M(g)$.

To begin our proofs, we note that (1) implies that

$$\operatorname{sgn}(\lambda^* * \mu)^\wedge(F) \subset \{z : |\arg z| \leqq 2\phi\}$$

where $\phi = \sin^{-1}(\theta/2) < \pi/4$. Thus the mean value $M(\hat{\lambda}\bar{\hat{\mu}})$ lies in the convex hull of

$$\{z : |\arg z| \leqq 2\phi, |z| \geqq \inf|\hat{\lambda}\bar{\hat{\mu}}|(F)\},$$

so

$$\begin{aligned}\operatorname{Re} M(\hat{\lambda}\bar{\hat{\mu}}) &\geqq (\inf|\hat{\lambda}\bar{\hat{\mu}}|(F))\cos 2\phi \\ &= (\inf|\hat{\lambda}\hat{\mu}|(F))(1-2\sin^2\phi) \\ &= (1-\theta^2/2)\inf|\hat{\lambda}\bar{\hat{\mu}}|(F).\end{aligned}$$

Since $M(\hat{\lambda}\bar{\hat{\mu}}) = \lambda^* * \mu\{0\}$ [5], we have x_j in G for which

$$\operatorname{Re}\lambda^* * \mu\{0\} = \operatorname{Re}\sum_j \overline{\lambda\{x_j\}}\,\mu\{x_j\} \geqq (1-\theta^2/2)\inf|\hat{\lambda}\hat{\mu}|(F) > 0$$

and consequently we deduce that (1) on F implies λ and μ cannot be mutually singular. But now let $\lambda_s = \lambda - \lambda_\mu$, the component of λ singular with respect to $|\mu|$. Then since $\eta = \frac{1}{2}(\sqrt{2}-\theta) < 1$, if our conclusion fails we have $\|\hat{\lambda}^{-1}\hat{\lambda}_\mu\|_\infty \leqq \eta < 1$, so

$$|\hat{\lambda}_s| \geqq |\hat{\lambda}| - |\hat{\lambda}_\mu| \geqq (1-\eta)|\hat{\lambda}|$$

and $|\hat{\lambda}_s|$ is also bounded away from zero on F. Thus

$$\left|\frac{\hat{\lambda}_s}{|\hat{\lambda}_s|} - \frac{\hat{\lambda}}{|\hat{\lambda}|}\right| \leqq \left|\frac{\hat{\lambda}_s}{|\hat{\lambda}_s|} - \frac{\hat{\lambda}_s}{|\hat{\lambda}|}\right| + \frac{|\hat{\lambda}_s - \hat{\lambda}|}{|\hat{\lambda}|} \leqq \frac{|\hat{\lambda}_s|\,\big||\hat{\lambda}| - |\hat{\lambda}_s|\big|}{|\hat{\lambda}_s||\hat{\lambda}|} + \frac{|\hat{\lambda}_\mu|}{|\hat{\lambda}|}$$

$$\leqq 2\|\hat{\lambda}^{-1}\hat{\lambda}_\mu\|_\infty < 2\eta,$$

whence

$$\begin{aligned}&|\operatorname{sgn}\hat{\lambda}_s - \operatorname{sgn}\hat{\mu}| \leqq 2\|\hat{\lambda}^{-1}\hat{\lambda}_\mu\|_\infty + |\operatorname{sgn}\hat{\lambda} - \operatorname{sgn}\hat{\mu}| \\ &\leqq \theta' = 2\|\hat{\lambda}^{-1}\hat{\lambda}_\mu\|_\infty + \theta < 2\eta + \theta = (\sqrt{2}-\theta) + \theta = \sqrt{2} \text{ on } F,\end{aligned}$$

and now the first part of our argument, applied in λ_s, μ and θ', shows λ_s and μ cannot be mutually singular, completing our proof of Theorem 1.

The proof of Theorem 2 is a refinement of the first part of the preceding argument. Noting again that (1) locates $(\lambda^* * \mu)^\wedge(F)$ in the cone $\{z : |\arg z| \leqq 2\phi\}$ we have $\operatorname{Re}\hat{\lambda}\bar{\hat{\mu}} \geqq |\hat{\lambda}\hat{\mu}|(1-\theta^2/2)$ on F; again we apply the mean M, but use the fact that the transform of a continuous measure lies in the ideal of functions of mean value zero:

$$(1-\theta^2/2)M(|\bar{\hat{\lambda}}_d\hat{\mu}_d|) = (1-\theta^2/2)M(|\bar{\hat{\lambda}}\hat{\mu}_d|) = (1-\theta^2/2)M(|\bar{\hat{\lambda}}\hat{\mu}|)$$

$$\leqq M(\operatorname{Re}\bar{\hat{\lambda}}\hat{\mu}) = \operatorname{Re} M(\bar{\hat{\lambda}}_d\hat{\mu}_d) \leqq |\lambda_d^* * \mu_d\{0\}| = \left|\sum_{x\in G} \overline{\lambda\{x\}}\mu\{x\}\right|$$

$$= \left|\sum \overline{\lambda_{\mu_d}\{x\}}\mu_d\{x\}\right| = \left|\sum \overline{\lambda_\mu\{x\}}\mu_d\{x\}\right| = |M(\bar{\hat{\lambda}}_\mu\hat{\mu}_d)|.$$

Now from the first, second, and last quantities, we obtain the inequalities

$$(1-\theta^2/2)\inf|\hat{\lambda}_d(F)|M(|\hat{\mu}_d|) \leqq \|\hat{\lambda}_\mu\|_\infty M(|\hat{\mu}_d|),$$

$$(1-\theta^2/2)\inf|\hat{\lambda}(F)|M(|\hat{\mu}_d|) \leqq \|\hat{\lambda}_\mu\|_\infty M(|\hat{\mu}_d|),$$

while we know $\mu_d \neq 0$ so $M(|\hat{\mu}_d|) > 0$. Thus $(1-\theta^2/2)\inf|\hat{\lambda}(F)|$ and $(1-\theta^2/2)\inf|\hat{\lambda}_d(F)|$ are bounded by $\|\hat{\lambda}_\mu\|_\infty$. In fact, however, the first of these is the smaller since $\hat{\lambda}_d(\Gamma) \subset \hat{\lambda}(F)^-$ by [3, cor. 4].

By exactly the proof of Theorem 2, we can obtain a version involving ordered n-tuples $\Lambda = (\lambda_1, \cdots, \lambda_n)$ of elements of $M(G)$, simply replacing absolute values by the euclidean norm.

THEOREM 2′. *Let $\Lambda = (\lambda_1, \cdots, \lambda_n)$ and $\Lambda' = (\lambda'_1, \cdots, \lambda'_n)$ be two ordered n-tuples of elements of $M(G)$ for which, for some $\delta > 0$ and $\theta < \sqrt{2}$, we have, for each compact $C \subset \Gamma$, some translate of C on which*

$$(1') \qquad |\hat{\Lambda}| \geqq \delta, \quad |\hat{\Lambda}'| \geqq \delta \quad \text{and} \quad \left|\frac{\hat{\Lambda}}{|\hat{\Lambda}|} - \frac{\hat{\Lambda}'}{|\hat{\Lambda}'|}\right| \leqq \theta.$$

Then if Λ_a is the n-tuple of measures $(\lambda_j)_{\lambda'_j}$, we have $\delta^{-1}\|\hat{\Lambda}_a\|_\infty \geqq (1-\theta^2/2)$.

(Here $\hat{\Lambda} = (\hat{\lambda}_1, \cdots, \hat{\lambda}_n)$ of course; one must only note that (1′) implies $\operatorname{Re}\Sigma(\lambda_j^* * \lambda'_j)^\vee \geqq |\hat{\Lambda}||\hat{\Lambda}'|(1-\theta^2/2)$.)

Although it is clear for (3) from our proof, we can also trace (2) back to overlap of the discrete parts of λ and μ. Indeed if (1) holds on F, the generator of an invariant filter in the sense of [3], then by its main result we have a net $\{\gamma_\delta\}$ in F for which $(\hat{\mu}\chi_F)_{\gamma_\delta} \to \hat{\mu}_d$ and $(\hat{\lambda}\chi_F)_{\gamma_\delta} \to \hat{\lambda}_d$ pointwise on Γ_0 (since $\hat{\mu}_d$, $\hat{\lambda}_d$ are the almost periodic components of $\hat{\mu}$, $\hat{\lambda}$); thus $|\operatorname{sgn}\hat{\mu}_d - \operatorname{sgn}\hat{\lambda}_d| \leqq \theta < \sqrt{2}$ on Γ follows from (1). Since, as we have noted, $\hat{\lambda}_d(\Gamma) \subset \hat{\lambda}(F)^-$ and $\hat{\mu}_d(\Gamma) \subset \hat{\mu}(F)^-$, we also have $\hat{\lambda}_d$ and $\hat{\mu}_d$ bounded away from 0, and so can conclude from Theorems 1 and 2 that

$$(2'') \qquad \|\hat{\lambda}_d^{-1}(\lambda_d)\hat{}_{\mu_d}\|_\infty \geqq \eta$$

and

$$(3'') \qquad \|(\lambda_d)^{\wedge}_{\mu_d}\|_\infty \geqq (1-\theta^2/2)\inf|\hat{\lambda}_d|.$$

But now we can use the almost immediately evident fact that $(\lambda_d)_{\mu_d} = (\lambda_\mu)_d$ along with our inclusion $\hat{\lambda}_d(\Gamma) \subset \hat{\lambda}(F)^-$ to obtain the conclusion of Theorem 2 from (3″), since by the latter $\|\hat{\lambda}_\mu\|_\infty \geqq \|(\lambda_\mu)^{\wedge}_d\|_\infty$ and $\inf|\hat{\lambda}_d| \geqq \inf|\hat{\lambda}(F)|$; thus

$$\|\hat{\lambda}_\mu\|_\infty \geqq \|(\lambda_\mu)^{\wedge}_d\|_\infty = \|(\lambda_d)^{\wedge}_{\mu_d}\| \geqq (1-\theta^2/2)\inf|\hat{\lambda}_d|$$
$$\geqq (1-\theta^2/2)\inf|\hat{\lambda}(F)|.$$

To obtain (2) itself in Theorem 1, we have for $\gamma \in \Gamma_0$, which is dense in Γ,

$$|\hat{\lambda}_d^{-1}(\gamma)(\lambda_d)^{\wedge}_{\mu_d}(\gamma)| = \lim|\hat{\lambda}^{-1}(\gamma+\gamma_\delta)\hat{\lambda}_\mu(\gamma+\gamma_\delta)|,$$

whence (2) follows from (2″).

Actually there is a formulation of our result for weakly almost periodic functions ϕ, ψ on Γ and their almost periodic components, ϕ_a, ψ_a, but the analogue of λ_μ seems less natural. Let

$$\mathrm{sp}_a(\phi) = \{g \in G = \Gamma^{\wedge}\colon M(\bar{g}\phi) \neq 0\}$$

be the almost periodic (or point) spectrum of ϕ, where M is invariant mean on the weakly almost periodic functions $W(\Gamma)$ on Γ.

THEOREM 2″. *For ϕ, $\psi \in W(\Gamma)$ and F the generator of an invariant filter on Γ, if ϕ and ψ are bounded from 0 on F and satisfy*

$$|\operatorname{sgn}\phi - \operatorname{sgn}\psi| \leqq \theta < \sqrt{2} \quad \textit{on } F$$

then

$$(2''') \qquad \sup\left\{\left\|\sum_{g\in E} M(\bar{g}\phi)g\right\|_\infty : E \textit{ finite} \subset \mathrm{sp}_a(\psi)\right\} \geqq (1-\theta^2/2)\inf|\phi(F)|.$$

This is easily seen to be our Theorem 2 when $\phi = \hat{\lambda}$, $\psi = \hat{\mu}$, but for a general weakly almost periodic function ϕ there is little reason to expect the left side of (2‴) to be finite, and indeed this is only the case when $\Sigma_{g\in \mathrm{sp}_a(\psi)}|M(\bar{g}\phi)| < \infty$. (In particular (2‴) does make an assertion (exactly) when the almost periodic component of ϕ has an absolute convergent Fourier series.)

In order to prove the result, note that as before, we know $|\operatorname{sgn}\phi_a - \operatorname{sgn}\psi_a| \leqq \phi$ on Γ and that it suffices to prove (2‴) for Γ, ϕ_a and ψ_a in place of F, ϕ, ψ since the left side is unchanged while the right side can only increase because $\phi_a(\Gamma) \subset \phi(F)^-$, again by [3]. So we may as well replace Γ by Γ^a, its Bohr compactification, and take G discrete. Now if (2‴) fails, then in $L^\infty(\Gamma)$ the net $E \to \phi_E = \Sigma_{\gamma\in E} M(\bar{g}\phi)g$ of trigonometric polynomials of norm $\leqq c <$

$(1-\theta^2/2)\inf|\phi(F)|$ must have a w^* cluster point ϕ_0, and $\|\phi_0\| \leqq c$. But now, since M corresponds to the Haar integral over Γ^a, $\operatorname{Re} M(\bar{\phi}_a\psi_a) = \lim \operatorname{Re} M(\bar{\phi}_E\psi_a) = \operatorname{Re} M(\bar{\phi}_0\psi_a)$ while we know

$$\operatorname{Re} M(\bar{\phi}_a\psi_a) \geqq (1-\theta^2/2) M(|\bar{\phi}_a\psi_a|)$$

so that

$$(1-\theta^2/2)\inf|\phi_a| M(|\psi_a|) \leqq \operatorname{Re} M(\bar{\phi}_0\psi_a) \leqq \|\bar{\phi}_0\| M(|\psi|)$$
$$\leqq c M(|\psi|) < (1-\theta^2/2)\inf|\phi(F)| M(|\psi_a|).$$

Since $\phi_a(\Gamma) \subset \phi(F)^-$, $\inf|\phi_a(\Gamma)| \geqq \inf|\phi(F)|$ so we have the desired contradiction.

The extent to which Theorem 2 can be applied can be made quite explicit: $|\hat{\lambda}|$ *and* $|\hat{\mu}| > \delta > 0$ *and* (1) *hold for some* $\theta < \sqrt{2}$ *on some generator of an invariant filter if and only if* λ_d *and* μ_d *are invertible and* $\nu = \lambda_d^{-1} * \mu_d$ *satisfies* $|\operatorname{sgn} \hat{\nu} - 1| \leqq \theta' < \sqrt{2}$ *on* Γ.

Once we note λ_d is invertible iff $\hat{\lambda}_d$ is, we have essentially seen "only if"; in the other direction if $|\operatorname{sgn} \hat{\nu} - 1| \leqq \theta'$ then $|\operatorname{sgn} \hat{\lambda}_d - \operatorname{sgn} \hat{\mu}_d| \leqq \theta'$, and we have only to note that [3, cor. 2] for $\delta > 0$

$$F_\delta = \{\gamma : |\hat{\mu}(\gamma) - \hat{\mu}_d(\gamma)|^2 + |\hat{\lambda}(\gamma) - \hat{\lambda}_d(\gamma)|^2 < \delta\}$$

generates an invariant filter, on which for $\delta = \delta_\varepsilon$ small, $|\operatorname{sgn} \hat{\lambda} - \operatorname{sgn} \hat{\mu}| \leqq \theta' + \varepsilon < \sqrt{2}$, by exactly the use of the triangle inequality employed in proving Theorem 1.

There is another variant of our results which should be noted.

COROLLARY 1. *Suppose* $\{m_\delta\}$ *is a net of probability measures on* Γ *which converges pointwise to the invariant mean* M *on* $W(\Gamma)$. *If* $|\hat{\lambda}|$ *and* $|\hat{\mu}|$ *are bounded away from* 0 *and*

$$m_\delta\{\gamma : |\operatorname{sgn} \hat{\mu}(\gamma) - \operatorname{sgn} \hat{\lambda}(\gamma)| \leqq \theta\} \to 1$$

where $\theta < \sqrt{2}$, *then*

$$\|\hat{\lambda}^{-1}\hat{\lambda}_\mu\|_\infty \geqq \frac{1}{2}(\sqrt{2} - \theta) \qquad \textit{and} \qquad \|\hat{\lambda}_\mu\|_\infty \geqq (1-\theta^2/2)\inf|\hat{\lambda}|.$$

For let ϕ denote the extension of the weakly almost periodic function $|\operatorname{sgn} \hat{\mu} - \operatorname{sgn} \hat{\lambda}|$ to the weakly almost periodic compactification Γ^w of $\Gamma[0]$, and choose $\psi \in C(\Gamma^w)$ with

$$\chi_{\{\phi \leqq \theta\}} \leqq \psi \leqq \chi_{\{\phi \leqq \theta + \varepsilon\}},$$

$\theta+\varepsilon<\sqrt{2}$; since $\{m_\delta\}$ converges w^* on $C(\Gamma^w)$ to m, the Haar measure of the least ideal Γ^a of Γ^w (which is the measure of Γ^w providing our invariant mean $M[0]$) we have

$$1=\lim m_\delta\{\phi\leqq\theta\}\leqq\lim\int\psi dm_\delta=\int\psi dm\leqq m\{\phi\leqq\theta+\varepsilon\}$$

so that $\phi\leqq\theta+\varepsilon$ everywhere on Γ^a. But that amounts precisely to the assertion that $|\operatorname{sgn}\hat{\mu}_d-\operatorname{sgn}\hat{\lambda}_d|\leqq\theta+\varepsilon$, so (2″) and (3) follow as before.

Thus for $G=\mathbf{R}$, it's enough to know that on each of a sequence of intervals I_n with $|I_n|\to\infty$ we have (1) holding $\eta_n|I_n|$ of the time, with $\eta_n\to 1$; more interestingly (2) and (3) follow if $(1/n^2)\operatorname{card}\{\gamma\in\{1,2,\cdots,n\}+\sqrt{2}\{1,2,\cdots,n\}:|\operatorname{sgn}\hat{\mu}(\gamma)-\operatorname{sgn}\hat{\lambda}(\gamma)|\leqq\theta\}\to 1$.

(Here our hypothesis for the corresponding normalized counting measures m_n holds by the following argument: for x in the subsemigroup $S=\mathbf{Z}_+ +\sqrt{2}\,\mathbf{Z}_+$ of $\mathbf{R}$ we evidently have $\|\delta_x*m_n-m_n\|\to 0$, and thus $\delta_x*m=m$ for any w^* cluster point m of $\{m_n\}$ on $\mathbf{R}^w$; since [3, 0] the least ideal $\mathbf{R}^a$ of $\mathbf{R}^w$ lies in the closure S^- of S in $\mathbf{R}^w$ and $\delta_x*m=m$ for any x in S, this says $\delta_x*m=m$ for any x in the least ideal $\mathbf{R}^a$, so that m must be the Haar measure of $\mathbf{R}^a$.)

Theorem 1 and the results of §2 were originally obtained as direct consequences of the following simple lemma.

LEMMA 1. *Suppose the identity of G does not lie in the closed support S_ν of $\nu\in M(G)$. Then* $\operatorname{Re}\hat{\nu}\geqq 0$ *implies* $\operatorname{Re}\hat{\nu}\equiv 0$.

To prove the lemma, let U be a symmetric neighborhood of 0 for which $(U-U)\cap S_\nu=\emptyset$. Then for $f=\chi_U*\chi_U$ we know f is a non-negative element of $L^1(G)$ while $\hat{f}\geqq 0$ lies in $L^1(\Gamma)$, and $f\nu=0$. Thus $0=(f\nu)^\wedge=\hat{f}*\hat{\nu}$ (by the inversion formula and Fubini), and so

$$0\equiv\operatorname{Re}\hat{f}*\hat{\nu}=\hat{f}*\operatorname{Re}\hat{\nu}.$$

But the continuous function $\operatorname{Re}\hat{\nu}\geqq 0$ while $\hat{f}\geqq 0$, so this implies $\operatorname{Re}\hat{\nu}\equiv 0$ on any translate of the support of $\hat{f}$, hence on all of Γ.

COROLLARY 2. *If $0\notin(S_\mu-S_\lambda)^-$ then* (1) *on Γ implies $\hat{\lambda}\hat{\mu}\equiv 0$.*

Indeed $(S_\mu-S_\lambda)^-$ supports the measure $\lambda^{*}*\mu$ and from (1)

$$|1-\operatorname{sgn}\bar{\hat{\lambda}}\operatorname{sgn}\hat{\mu}|=|1-\operatorname{sgn}(\lambda^{*}*\mu)^\wedge|\leqq\theta<\sqrt{2}$$

so $(\lambda^{*}*\mu)^\wedge(\Gamma)$ lies in a proper subcone of the right half plane in $\mathbf{C}$. Now for $\varepsilon>0$ small we have $\operatorname{Re}e^{\pm i\varepsilon}\bar{\hat{\lambda}}\hat{\mu}\geqq 0$ so that $\operatorname{Re}e^{\pm i\varepsilon}\bar{\hat{\lambda}}\hat{\mu}\equiv 0$ by the lemma, whence $\bar{\hat{\lambda}}\hat{\mu}\equiv 0$.

COROLLARY 3. *Suppose* $0 \not\in (S_\mu - A)^-$, $\mu \neq 0$, *and* $\{\mu_n\}$ *is a sequence of measures carried by* A. *If, for almost all* γ *in* $\Gamma \setminus \hat{\mu}^{-1}(0)$ *we have all cluster points of* $\{\bar{\hat{\mu}}_n \hat{\mu}(\gamma)\}$ *lying in the sector* $|\arg z| \leqq \pi/2 - \varepsilon$, *while* $\underline{\lim} |\hat{\mu}_n(\gamma)| > 0$ *for* γ *in a subset of* $\Gamma \setminus \hat{\mu}^{-1}(0)$ *of positive measure, then* $\|\hat{\mu}_n\| \to \infty$.

In case $G = \mathbf{R}$ and A is an infinite compact set then we can uniformly approximate $\hat{\mu}$ on compacta by $\{\hat{\mu}_n\}$ with μ_n in $M(A)$, indeed so that $\sup |\hat{\mu}_n - \hat{\mu}|([-n, n]) \leqq 1/n$: otherwise some non-zero measure λ on $[-n, n]$ would be orthogonal to $M(A)^\wedge$, whence we obtain non-zero entire function $\hat{\lambda}$ vanishing on A. Taking $\hat{\mu}$ bounded away from zero we have $\|\hat{\mu}_n\| \to \infty$ by the Corollary.

For its proof, suppose we have a bounded subsequence of $\{\|\hat{\mu}_n\|_\infty\}$ (which we take as the whole sequence). As in the proof of the lemma we have $0 = \hat{f} * \operatorname{Re} \bar{\hat{\mu}}_n \hat{\mu}$, so since $\{\|\mu_n\|_\infty\}$ is bounded we obtain from Fatou that

$$0 = \underline{\lim}\, \hat{f} * \operatorname{Re} \bar{\hat{\mu}}_n \hat{\mu}(\gamma) \geqq \hat{f} * \underline{\lim} \operatorname{Re} \bar{\hat{\mu}}_n \hat{\mu}(\gamma).$$

Since the last $\underline{\lim}$ is $\geqq 0$ by hypothesis we conlude $\underline{\lim} \operatorname{Re} \bar{\hat{\mu}}_n \hat{\mu} = 0$ a.e. But this is inconsistent with our hypotheses on cluster values of $\bar{\hat{\mu}}_n \hat{\mu}$ which together imply $\underline{\lim} \operatorname{Re} \bar{\hat{\mu}}_n \hat{\mu} > 0$ on a set of positive measure.

§2. The original purpose of Lemma 1 was to extend Rajchman's theorem for certain measures[†], replacing (0) by the hypothesis that the set of cluster values of $\hat{\mu}$ at $+\infty$ lies in a closed cone K (with vertex 0) in $\mathbf{C}$, i.e., that

$$\mathrm{cl}(\hat{\mu}, +\infty) \subset K, \quad K \neq \mathbf{C}. \tag{3}$$

As we shall see (Theorems 3 and 5) this can be done in a couple of ways.

The set of Fourier–Stieltjes transforms satisfying (3) is translation invariant, and closed under the formation of linear combinations with positive coefficients; alternatively, the corresponding set of measures on T is invariant under multiplication by P_+, the set of positive definite trigonometric polynomials (precisely those trigonometric polynomials with positive coefficients). So to conclude that any measure $\nu \ll |\mu|$ also satisfies (3) it is sufficient to know P_+ is dense in $L^1(|\mu|)$: (repeating the argument now used[††] to prove Rajchman's theorem) then $p_n \mu \to \nu$ in norm for a sequence $\{p_n\}$ in P_+, and since $(p_n\mu)^\wedge$ satisfies (3) for each n it is clear $\hat{\nu}$ must. In fact since we also have p_n in P_+ for which $p_n \mu \to e^{i\theta}\nu$ we conclude that

[†] For a full (and much deeper) extension, see [2].

[††] I learned of this argument from Frank Forelli.

$$\mathrm{cl}(\hat{\nu}, +\infty) \subset e^{-i\theta}K$$

for each θ, and thus that our cluster set is in fact $\{0\}$ itself. Evidently we can now assert (for those μ in $M(T)$ for which P_+ is dense in $L^1(|\mu|)$) that (3) implies $\mathrm{cl}(|\mu|^\wedge, +\infty) = \{0\}$ hence that $|\mu|^\wedge \in C_0$, whence $\hat{\nu} \in C_0$ for each $\nu \ll \mu$ by the same argument.

We thus arrive at the question of when P_+ is dense in $L^1(|\mu|)$, which divides naturally in two: when the closure P_+^- is real linear, hence precisely the real span of Γ, and when that real span is complex linear.

LEMMA 2. *Let G be a locally compact abelian group, $\mu \in M(G)$, and $P_+ = \{\Sigma\, a_i\gamma_i : a_i \geqq 0,\ \gamma_i \in \Gamma\}$ be the set of positive definite trigonometric polynomials on G. Then* (1) *the closure P_+^- in $L^1(|\mu|)$ of P_+ is a real linear subspace iff* (2) $\mathrm{Re}(g\mu)^\wedge \geqq 0$ *implies* $\mathrm{Re}(g\mu)^\wedge \equiv 0$ *for g in $L^\infty(|\mu|)$.*

As mentioned the subspace is $(P_+ - P_+)^-$, the real linear span of Γ; since (2) asserts that any continuous real linear functional which supports the cone P_+^- is identically zero, the equivalence is clear. (As we shall see later $P_+^- = (P_+ - P_+)^-$ itself yields a variant of Rajchman's result (Theorem 5).)

Thus $0 \notin S_\mu$ is sufficient to guarantee (1) by Lemma 1, but it is hardly necessary. Indeed for a compact group G if $|\mu|(U) \to 0$ rapidly enough as the neighborhood U of 0 shrinks to 0 we can obtain (1): as Katznelson has pointed out to me it's sufficient that we have, say, an approximate identity† of trigonometric polynomials $\{p_\delta\}$ in P_+ for which $\|p_\delta\mu\| \to 0$; then $\hat{p}_\delta \to 1$ pointwise and so, for any trigonometric polynomial p with real coefficients, we eventually have $2\|\hat{p}\|_\infty p_\delta + p \in P_+$ while $2\|\hat{p}\|_\infty \|p_\delta\mu\| \to 0$, so $p \in P_+^-$. Thus in particular for $G = T = (-\pi, \pi]$ it suffices to know $\int_0^{1/n} |\mu|(-t, t)dt = o(n^{-2})$ since this is precisely this condition for a familiar approximate identity, viz., that $\int_{-1/n}^{1/n} n(1 - n|t|)|\mu|(dt) = \int u_n(t)|\mu|(dt) \to 0$, while the u_n are uniformly approximated by elements of P_+ because $\hat{u}_n \geqq 0$ lies in $L^1(\mathbf{Z})$. An exact analytic formulation of the restriction on $|\mu|$ implict in (1) is not clear. (Note that for G compact Katznelson's observation yields an easier proof of Lemma 1: for $\mathrm{Re}(p\mu)^\wedge \geqq 0$, $p \in P_+$, and choosing p_n so that $p_n\mu \to -\mu$ we obtain $-\mathrm{Re}\,\hat{\mu} \geqq 0$.)

† In fact we only need a net $\{p_\delta\}$ in P_+ for which $\|p_\delta\mu\| \to 0$ and $\underline{\lim}\, \hat{p}_\delta(\gamma) \geqq 1$, all γ, and this is also necessary: for $E \subset \Gamma$ finite and $\varepsilon > 0$ we must have $p = p_{E,\varepsilon} \in P_+$ with $\hat{p} \geqq \chi_E$ and $\|p\mu\| < \varepsilon$; otherwise for some E and ε, $p \in P_+$, $\hat{p} \geqq \chi_E$ imply $\|p\mu\| \geqq \varepsilon$, and for $p_0 \in P_+$ with $\hat{p}_0 = \chi_E$ this says the convex set $p_0 + P_+$ has L_1 distance $\geqq \varepsilon$ from 0; so $\mathrm{Re}\int g(p_0 + p)d\mu \geqq 1$ for some $g \in L^\infty(|\mu|)$ and all $p \in P_+$, yielding a non-vanishing supporting functional for P_+.

As an example where this fails although $\mu\{0\} = 0$, on $T = (-\pi, \pi]$ take $\mu(dt) = |t|^{-1/2}\max(1 - |t|, 0)dt$, which has $\hat{\mu} \geqq 0$ as one shows in one proof of Polya's theorem [1, p. 183].

The second part of our question is when $(P_+ - P_+)^-$ is complex linear, and here our answer is simple and complete.

LEMMA 3. *The closure* $(P_+ - P_+)^-$ *in* $L^1(|\mu|)$ *of the real linear span of* Γ *is complex linear iff* μ *and* μ^* *are mutually singular.*

Since the real linear span of Γ is complex linear iff it is $L^1(|\mu|)$ itself we have this iff every continuous linear functional on the *real* Banach space $L^1(|\mu|)$ orthogonal to $P_+ - P_+$ (or to Γ) vanishes identically, i.e., iff $\operatorname{Re}\int h\gamma d\mu = 0$, all $\gamma \in \Gamma$ implies $h = 0$ in $L^\infty(|\mu|)$. The last hypothesis says precisely that $(h\mu)^\wedge$ has purely imaginary values, so we have just to know $(h\mu)^\wedge$ real valued implies $h = 0$ in $L^\infty(|\mu|)$ (or $h\mu = 0$) for $h \in L^\infty(|\mu|)$. Because $(h\mu)^\wedge$ is real valued exactly when $h\mu = (h\mu)^*$ we must have $h\mu = 0$ if μ and μ^* are mutually singular since $(h\mu)^* \ll \mu^*$. Conversely, if μ and μ^* are not singular we have an $f \in L^1(|\mu|)$ with $0 \neq f\mu \ll \mu^*$, so $f\mu = g\mu^*$; multiplying this last equality by an appropriate characteristic function we can assume both f and g are bounded, so $(f\mu)^* = g^*\mu$ lies in $L^\infty(|\mu|)\mu$ (where $g^*(x) = \overline{g(-x)}$). But now one of the elements $f\mu + (f\mu)^*$, $(1/2i)(f\mu - (f\mu)^*)$ is non-zero, lies in $L^\infty(|\mu|)\mu$, and has a real transform, completing our proof of Lemma 3.

For $G = T$ we thus have the following variant of Rajchman's result, by the argument preceding Lemma 2.

THEOREM 3. *Suppose* K *is a closed cone in* $\mathbf{C}$, $K \neq \mathbf{C}$, *with vertex* 0, *and* μ *is a measure on the circle group* $T = (-\pi, \pi]$ *singular with respect to its reflection* μ^* *and having*

$$\int_0^{1/n} |\mu|(-t, t)dt = o(n^{-2}).$$

Then

$$\operatorname{cl}(\hat{\mu}, +\infty) \subset K \tag{3}$$

implies $\hat{\nu} \in C_0(\mathbf{Z})$ *for any* $\nu \ll \mu$.

The only structure in $\mathbf{Z}$ involved with the translation invariance implicit in (3) can be formulated generally: for closed subsets S, S_0 of Γ *let us say* S_0 *translates into* S *modulo compacta if* $(\gamma + S_0)\backslash S$ *is relatively compact for each* γ *in* Γ. Further, let $\operatorname{cl}(\hat{\mu}, S)$ be the set of cluster values of $\hat{\mu}$ along S, i.e., the set of limits of nets $\{\hat{\mu}(\gamma_\delta)\}$ where $\gamma_\delta \in S$ and $\gamma_\delta \to \infty$ in Γ. Then if S_0 translates into S modulo compacta and $\operatorname{cl}(\hat{\mu}, S) \subset K$ we have $\operatorname{cl}((p\mu)^\wedge, S_0) \subset K$ for any $p \in P_+$: for if $z \in \operatorname{cl}((p\mu)^\vee, S_0) = \operatorname{cl}(\Sigma a_j \hat{\mu}_{\gamma_j}, S_0)$ then $z = \lim \Sigma_{j=1}^n a_j \hat{\mu}(\gamma_\delta - \gamma_j)$ where $a_j \geqq 0$, $\gamma_\delta \in S_0$ and $\gamma_\delta \to \infty$ in Γ, and we can assume $\hat{\mu}(\gamma_\delta - \gamma_j) \to z_j$ for each j; but then

$\gamma_\delta - \gamma_j \to \infty$ in Γ so $\gamma_\delta - \gamma_j \in S$ for $\delta \geqq \delta_j$ as well, whence $z_j \in \mathrm{cl}(\hat{\mu}, S) \subset K$, and $z = \sum a_j z_j \in K$.

Now if P_+ is dense in $L^1(|\mu|)$ we can again use the argument of Rajchman's theorem to show $\mathrm{cl}(\hat{\nu}, S_0) = \{0\}$, $\nu \ll \mu$; since $\mathrm{cl}(|\mu|^\wedge, -S_0) = \mathrm{cl}(|\mu|^\wedge, S_0) = \{0\}$ follows we have $\mathrm{cl}(\hat{\nu}, -S_1) = \{0\}$ if S_1 translates into S_0 modulo compacta. Thus

THEOREM 4. *Suppose G and Γ are dual locally compact abelian groups and $\mu \in M(G)$ and μ^* are mutually singular, while $0 \in G$ does not lie in the closed support of μ. If*

$$\mathrm{cl}(\hat{\mu}, S) \subset K, \tag{4}$$

a closed cone properly contained in $\mathbf{C}$, while S_0 translates into S modulo compacta, then $\mathrm{cl}(\hat{\nu}, S_0) = \{0\}$ for $\nu \ll \mu$; and if S_1 translates in S_0 modulo compacta as well then $\mathrm{cl}(\hat{\nu}, -S_1) = \{0\}$.

When we only know P_+^- forms a real subspace we can obtain a variant of Rajchman's therorem in which our conclusion exactly parallels our hypothesis.

THEOREM 5. *Suppose $\mu \in M(T)$ has (say) $\int_0^{1/n} |\mu|(-t, t)dt = o(n^{-2})$, and suppose K is a proper cone in $\mathbf{C}$, or, if $K = \{z : \mathrm{Re}\, e^{i\theta} z \geqq 0\}$, that $\mu \ll e^{i\theta}\mu + e^{-i\theta}\mu^* = \mu_\theta$. If*

$$\mathrm{cl}(\hat{\mu}, +\infty) \subset K \tag{3}$$

then the same is true with $\hat{\mu}$ replaced by $\hat{\nu}$, $\nu \ll \mu$, and $\mathrm{cl}(\hat{\mu}, -\infty) \subset K$. Finally, if K is a proper cone, all our cluster sets reduce to $\{0\}$, and $\hat{\nu} \in C_0(\mathbf{Z})$ all $\nu \ll \mu$.

We shall use the conditional weak compactness of the set of measures $\gamma\mu$, $\gamma \in \Gamma$, along with the fact that any weak cluster point must be of the form $g\mu$, $g \in L^\infty(|\mu|)$. Indeed we have only to note that any weak cluster point $g\mu$ of a net $\{\gamma_\delta \mu\}$, where $\gamma_\delta \to +\infty$ in $\mathbf{Z} = \Gamma$, must have $\mathrm{Re}(e^{i\theta} g\mu)^\wedge(\gamma) \geqq 0$ for any θ in $\mathbf{R}$ with $\mathrm{Re}\, e^{i\theta} k \geqq 0$ all $k \in K$; since this is true for all γ in Γ, by Lemma 2 we know $\mathrm{Re}(e^{i\theta} g\mu)^\wedge \equiv 0$. But because $e^{i\theta} g\mu$ is any weak cluster point of any net $\{\gamma_\delta e^{i\theta}\mu\}$ with $\gamma_\delta \to +\infty$ we have

$$\mathrm{cl}(\mathrm{Re}(e^{i\theta}\mu)^\wedge, +\infty) = 0,$$

so $\hat{\mu}_\theta(\gamma) \to 0$ as $\gamma \to +\infty$ for the measure $\mu_\theta = e^{i\theta}\mu + (e^{i\theta}\mu)^*$. By Rajchman's theorem the same is true for each $\nu \ll \mu_\theta$ and in particulår for $|\mu_\theta|$, so we have the same behavior at $-\infty$ (for $|\mu_\theta|$ and therefore) for μ_θ. Now if K is proper we have $|\mu|\{d\mu^*/d\mu = -e^{2i\theta}\} = 0$ for a dense set of those θ yielding supporting functionals for K, and this insures $\mu \ll \mu_\theta$ (for $e^{i\theta}\mu(E) + e^{-i\theta}\mu^*(E) = 0$ for all

$E \subset E_0$ implies $d\mu^*/d\mu = -e^{2i\theta}\,|\mu|$-a.e. on E_0). Since we are assuming $\mu \ll \mu_\theta$ if K is not proper, we now have

$$\mathrm{cl}(\mathrm{Re}\, e^{i\theta}\hat{\nu}, \pm\infty) \geqq 0$$

for all $\nu \ll \mu$, for sufficiently many θ to insure $\mathrm{cl}(\hat{\nu}, \pm\infty) \subset K$, and we are done once we note (for our final assertion) that for K proper, 0 is the only element on which all supporting functionals vanish.

Precisely the same sort of argument yields an alternate version of Theorem 4.

THEOREM 6. *Suppose G and Γ are dual locally compact abelian groups and $\mu \in M(G)$ does not have the identity of G in its closed support while*

$$\mathrm{cl}(\hat{\mu}, S) \subset K, \tag{5}$$

where K is proper, or if $K = \{z : \mathrm{Re}(e^{i\theta}z) \geqq 0\}$, $\mu \ll \mu_\theta = e^{i\theta}\mu + e^{-i\theta}\mu^*$. If S_0 translates into S modulo compacta then $\mathrm{cl}(\hat{\nu}, S_0) \subset K$ for any $\nu \ll \mu$; if S_1 translates into S_0 modulo compacta as well, then $\mathrm{cl}(\hat{\nu}, -S_1) \subset K$. Finally, when K is proper, all our cluster sets reduce to $\{0\}$.

(We have to note that here we obtain $\hat{\mu}_\theta(\gamma) = (e^{i\theta}\mu + e^{-i\theta}\mu^*)^\wedge(\gamma) \to 0$ as $\gamma \to \infty$ in S, or $\mathrm{cl}(\hat{\mu}_\theta, S) = \{0\}$, so we have $\mathrm{cl}(p\mu_\theta, S_0) = \{0\}$ with p *any* trigonometric polynomial, and the argument of Rajchman's theorem does apply.)

There are other variants in which the same sort of argument works. For example

THEOREM 7. *Suppose f is an even bounded continuous function on Γ for $f - f_\gamma \in C_0(\Gamma)$ for each $\gamma \in \Gamma$. If $0 \notin S_\mu$ and either $K = \{0\}$ or μ and μ^* are mutually singular, while S_0, S_1 and S are as in Theorem 6, then*

$$\mathrm{cl}(f\hat{\mu}, S) \subset K \tag{5'}$$

implies $\mathrm{cl}(f\hat{\nu}, S_0) \subset K$ and $\mathrm{cl}(f\hat{\nu}, -S_1) \subset K$ for all $\nu \ll \mu$.

We need only note that here $(f\hat{\mu})_\gamma = f\hat{\mu}_\gamma + (f_\gamma - f)\hat{\mu}_\gamma$ and $f_\gamma - f \in C_0(\Gamma)$ show $\mathrm{cl}(f\hat{\mu}_\gamma, S) = \mathrm{cl}((f\hat{\mu})_\gamma, S)$, whence (5′) allows us to conclude that $\mathrm{cl}(f(p\mu)^\wedge, S_0) \subset K$, $p \in P_+$ or, if $K = \{0\}$, for all p; since our hypotheses insures that P_+ is dense in $L^1(|\mu|)$ if $K \neq \{0\}$, in either case we obtain $\mathrm{cl}(f\hat{\nu}, S_0) \subset K$ since $\nu \to f\hat{\nu}$ is continuous from measure norms to sup norms, and we are done.

As illustrated in the proof of Theorems 5 and 6, the fact that $\Gamma\mu$ has weakly compact weak closure allows us to translate statements about cluster values of $\hat{\mu}$ into statements concerning values of certain Fourier–Stieltjes transforms (specifically of limits ν of nets $\{\gamma_\delta\mu\}$ in the weak topology, which coincides with the topology of pointwise convergence on Γ on the weak closure of $\Gamma\mu$). For example for $G = \mathbf{R}$ if we set

$$E_{+\infty}(\mu) = \bigcap_{r>0} \{\gamma\mu : \gamma > r\}^{-}$$

(where the bar denotes weak closure) and let $SE_{+}(\mu)$ denote its closed (hence weakly closed) span then

$$E_{+\infty}(|\mu|) \subset SE_{+\infty}(\mu)$$

by the argument yielding Rajchman's theorem, so $E_{-\infty}(\mu) \subset SE_{+\infty}(\mu)$. (In case $0 \notin S_\mu$ and μ and μ^* are mutually singular we can replace $SE_{+\infty}(\mu)$ everywhere by the obvious closed cones; thus if say $E_{+\infty}(\mu) \subset C = \{\lambda \ll \lambda_0 : \hat{\lambda} \geqq 0\} = \Gamma C$ so is $E_{+\infty}(|\mu|)$.)

One can concoct some involved applications of Theorem 6. For example suppose $0 \notin S_\mu$, $\mu \in M(\mathbf{R}^n)$ and μ is equivalent to a rotationally invariant measure λ, and $\mathrm{cl}(\hat{\mu}, S)$ lies in a proper cone, where S is the union of a sequence of balls $B_j = B(x_j, r_j)$ tending to ∞ with radii $r_j \to \infty$, while $(t, \infty) \subset \bigcup_j (|x_j| - \theta r_j, |x_j| + \theta r_j)$ for some $t \in \mathbf{R}$ and $\theta < 1$. Then using $S_0 = \bigcup_j B(x_j, \theta r_j)$ we can conclude that $\mathrm{cl}(\hat{\lambda}, S_0) = \{0\}$, so $\hat{\lambda} \in C_0(\mathbf{R}^n)$ since $\hat{\lambda}$ is radial, and thus $\hat{\mu} \in C_0(\mathbf{R}^n)$.

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